Seat No. : ________________

JA-105

January-2021

B.Sc., Sem.-III

201 : Mathematics

(Advanced Calculus – I)

Time : 2 Hours] **[Max. Marks : 50**

Instructions : (1) Attempt any **three** questions from **Q-1** to **Q-8**.

(2) **Q-9** is compulsory.

(3) Notations are usual, everywhere

(4) Figures to the right indicate marks of the question/sub-question.

1. (a) Define limit of function of two variables. Use the definition to find

$$\lim_{(x,y)\to(1,1)} \frac{x^2+y^2}{xy}.$$ 7

(b) Discuss the continuity of following functions at given point : 7

(1) $f(x, y) = \begin{cases} \dfrac{x^2-y^2}{x+y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ at point (0, 0)

(2) $f(x, y) = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at point (0, 1)

2. (a) Define iterated limits. Find the iterated limit for 7

$f(x, y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 2 & \text{if } (x, y) = (0, 0) \end{cases}$ at point (0, 0)

(b) Evaluate the following limit if exists 7

(1) $\lim_{(x,y)\to(0,0)} f(x, y)$ where

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 2 & \text{if } (x, y) = (0, 0) \end{cases}$$

(2) $\lim_{(x,y)\to(0,0)} f(x, y)$ where

$$f(x, y) = \begin{cases} \dfrac{\sin(x + y)}{x + y} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

3. (a) State and prove Young's theorem. 7

(b) Find $f_{xx}(0, 0)$, $f_{yy}(0,0)$, $f_{yx}(0, 0)$ and $f_{xy}(0, 0)$ for the function 7

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

4. (a) State and prove Schartz's theorem. 7

(b) Discuss the differentiability of the following functions : 7

(1) $f(x, y) = \begin{cases} \dfrac{x^3 y^3}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$ at point (0, 0)

(2) $f(x, y) = x^2 + y^2$ at point (0, 0)

5. (a) If $u = \phi(H)$ is function of a homogenous function $H = f(x, y)$ of degree m whose partial derivatives of second order exists, then 7

(1) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = m\dfrac{F(u)}{F'(u)}\ F'(u)\ (\neq 0) = G(u)$ (say)

(2) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = G(u)\,(G'(u) - 1)$

Where $H = f(x, y) = F(u)$.

(b) (1) If $f(x, y) = \sqrt{x^2 - xy}$, then prove that 7

$$x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 0$$

(2) Find the extreme values of $f(x, y) = x^3 + y^3 - 3axy$.

6. (a) State and prove Euler's theorem for homogenous function. 7

(b) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, then prove that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$. 7

7. (a) Find the radius of curvature of a curve $r = f(\theta)$ i.e. in polar equations. 7

(b) State and prove Taylor's theorem for the function of two variables. 7

8. (a) (1) Expand $f(x, y) = e^{ax} \sin by$ in the power of x and y. 7

(2) Find the radius of curvature of a curve $x^2 + y^2 = a^2$.

(b) Find the radius of curvature of parabola $r = a(1 - \cos \theta)$. 7

9. Attempt any **four** in short : 8

(a) Define multiple point and double point.

(b) Define conjugate point.

(c) Define harmonic function.

(d) If $u = e^{xy}$, then find $\frac{\partial^2 u}{\partial x \partial y}$.

(e) Find the degree of the homogenous function $z = \frac{x^{21/3} + x^{7/2} y^{7/2}}{x^5 + y^5}$.

(f) Give one example of function of two variables which is discontinuous at point (1, 1).