Seat No. : _______________

MC-115

March-2018

B.Sc., Sem.-V

CC-302 : Mathematics
(Analysis-I)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (i) There are **5** questions in this paper. **All** questions are compulsory.
(ii) Figures to the right indicate full marks of the question/sub-question.
(iii) Notations used in this question paper carry their usual meaning.

1. (a) Prove : **7**
(i) Show that the set $N \times N$ is countable.
(ii) Suppose that A and B are non-empty subsets of R that satisfy the property : $a \leq b$ for all $a \in A$ and all $b \in B$. Then prove that $\sup A \leq \inf B$.

OR

Let $f(x) = \frac{1}{x^2}$, $x \neq 0$, $x \in R$. (i) Determine the direct image f(E) where $E = \{x : 1 \leq x \leq 2\}$. (ii) Determine the inverse image $f^{-1}(G)$ where $G = \{x : 1 \leq x \leq 4\}$.

(b) Prove that there is a real number x such that $x^2 = 2$. **7**

OR

Let S be a subset of R which is bounded above and $a \in R$. Then let $a + S = \{a + s : s \in S\}$. Prove that $\sup (a + S) = a + \sup S$.

2. (a) Show by definition that the sequence $\{x_n\}$ defined by $x_n = \frac{3n+4}{2n}$ is convergent. For $\in = 0.001$, find the smallest positive integer n which satisfies the condition of definition. **7**

OR

Suppose that $\lim_{n \to \infty} x_n = a$ and $\lim_{n \to \infty} y_n = b$. If $b \neq 0$ $y_n \neq 0$ for all $n \in N$, then prove that the sequence $\left\{\frac{x_n}{y_n}\right\}$ is convergent, and $\lim_{n \to \infty} \frac{x_n}{y_n} = \frac{a}{b}$.

(b) If $S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$, then prove that $2 < S_n < 3$. And hence prove that it is convergent. **7**

OR

Prove that a is a cluster point of sequence $\{x_n\}$ iff for every $\in > 0$ and any $n_0 \in N$, there exists $n \geq n_0$ such that $| x_n - a | < \in$.

3. (a) State and prove Extreme Value Theorem (EVT). **7**

OR

Prove that a real valued function f is continuous at a if and only if for every sequence $\{x_n\}$ in D_f with $x_n \to a$, $f(x_n) \to f(a)$.

(b) Define removable discontinuity, jump discontinuity and 2nd type of discontinuity.

For $f(x) = \dfrac{(x-1)\,|x-2|}{x^2 - 3x + 2}$, discuss the types of discontinuity of f. **7**

OR

Determine whether the following functions are uniformly continuous or not ?

(i) $f(x) = \dfrac{1}{x}$ on (0, 1).

(ii) $f(x) = \dfrac{1}{x}$ on (a, b), a > 0.

4. (a) State and prove Mean Value Theorem. Verify it for $f(x) = \log x$ in [1, e]. **7**

OR

Suppose that f is defined on the open interval I and has a local maximum or local minimum at $c \in I$. Then if f is differential at c, prove that f'(c) must equal zero. Discuss its application on $f(x) = \sin x$.

(b) Evaluate : **7**

(i) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{\tan x - x}$

(ii) $\lim\limits_{x \to 0} \left(\dfrac{1}{x}\right)^{(1 - \cos x)}$

OR

State and prove L' Hospital's first rule.

5. Attempt any **SEVEN** of the following in short : **14**

(1) Give an example (if exists) of a set, which is superset of Q but countable.

(2) Give an example of a set whose lub does not exist but glb exists and belongs to the set.

(3) Find limit superior and limit inferior of the sequence $a_n = \sin \dfrac{n\pi}{4}$.

(4) Find cluster points of sequence $b_n = -n + (-1)^n$.

(5) Give an example of a function which is continuous but not uniformly continuous.

(6) Let $A = \{x \in R : x \neq 1\}$ and $B = \{y \in R : y \neq 2\}$; define $f : A \to B$, $f(x) = \dfrac{2x}{x-1}$ for all $x \in A$. Show that f is injective.

(7) Apply Rolle's theorem (if applicable) to $f(x) = \sin x$ on $[0, \pi]$ and find c as prescribed by the theorem.

(8) Evaluate $\lim\limits_{x \to 0} \dfrac{2 \sin x - \sin 2x}{x^3}$

(9) State Cauchy's Mean Value Theorem.
