Seat No. : _______________

MD-108

March-2018

B.Sc., Sem.-V

CC-303 : Mathematics
(Complex Variables and Fourier Series)

Time : 3 Hours] **[Max. Marks : 70**

Instructions : (i) **All** the questions are compulsory.
(ii) Each question is of **14** marks

1. (a) Define trigonometric and hyperbolic functions in C. Show that $| \sin z |^2 + | \cos z |^2 = ch2y$; $z \in C$. Also, express $\sqrt{3} - i$ in the exponential form. **7**

OR

State and prove De Moivre's theorem and hence solve the equation $z^5 - 1 = 0$; $z \in C$.

(b) Define the convergence of the sequence and series of complex numbers. **7**

If $z_n = a_n + ib_n$; $n = 1, 2, 3,...$ and $S = A + iB$, then prove that $\sum_{n=1}^{\infty} z_n = S$ if and only if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$.

OR

In the system C of complex numbers prove :

(i) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

(ii) $| z_1 z_2 | = | z_1 | | z_2 |$

Also solve the equation $z^2 + z + 1 = 0$

2. (a) Define limit of a complex function, harmonic function and analytic function . If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in the domain D, then derive Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ stating necessary conditions and verify the same for the function $f(z) = \sin h\, z$ where, $z = x + iy \in C$. **7**

OR

Define Entire function. Obtain the harmonic conjugate of $\frac{1}{2} \log (x^2 + y^2)$ and find the corresponding analytic function in terms of z. Would it be entire ? Justify.

(b) If the function $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic in the whole complex plane except at the origin, then derive $r^2v_{rr} + rv_r + v_{\theta\theta} = 0$. Verify the derived result for the function $\frac{1}{z}$. **7**

OR

Show that the function $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$; $(x, y) \neq (0, 0)$

$= 0$; $(x, y) = (0, 0)$

is not analytic at $z = 0$ even if f(z) satisfies Cauchy-Riemann equations at the origin.

3. (a) Define Conformal mapping. Prove that an analytic function f(z) is conformal at z_0 if and only if $f'(z_0) \neq 0$. **7**

OR

Define linear fractional transformation. Obtain the image of the curve $| z - 3 | < 2$ under the Mobius transformation $w = \frac{iz + 1}{z + i}$.

(b) Find the image of the strip $1 \leq x \leq 2$, y is real, under the mapping $w = \frac{1}{z}$, $z \neq 0$. **7**

OR

Find the critical (non-conformal) points and the angle of rotation of the mapping $w = z^3 - 3z^2 - 6z - 11$ at the point 2 + i. Also, obtain the bilinear transformation that maps 1, 0, –1 onto –1, 1, 0.

4. (a) If f(x) is Riemann integrable in $(-\pi, \pi)$, then prove that the series $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges, where a_n and b_n are the Fourier co-efficients of f(x). **7**

OR

Define the Fourier series for the function f and obtain the Fourier series expansion for the function $f(x) = x \sin x$, $-\pi < x < \pi$.

(b) Find the Fourier series for the function defined as $f(x) = x$ in $(0, \pi)$ and $f(x) = 2\pi - x$ in $(\pi, 2\pi)$. **7**

OR

Find the Fourier series expansion of the function $f(x) = x + x^2$ in $[-\pi, \pi]$.

5. Attempt any **Seven** in Short : **14**

(i) Identify the curve : $|z - i| = 1$.

(ii) Write the derivative of the functions sinh z, log z with respect to $z = x + iy \in C$.

(iii) Find the angle of rotation of the $f(z) = 1/z$ at the point $2 - i$.

(iv) Is the function $f(z) = \sin(\bar{z})$ analytic in the domain D ? Justify.

(v) 'If $u(x, y)$ and $v(x, y)$ are harmonic conjugate of each other then both are constant', justify.

(vi) Find the singular points of $z^2 + 2z - 1/(z-1)(z^2 - 7z + 12)$.

(vii) Obtain $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$ for all m, n = 0, 1, 2, . . .

(viii) State Bessel's inequality.

(ix) Find the image of the line $y = x - 1$ under the linear transformation $w = z + 2$.