Seat No. : _______________

DC-120

December-2018

M.Sc., Sem.-I

403 : Mathematics
(Complex Analysis – I)

Time : 2:30 Hours] **[Max. Marks : 70**

1. (a) (i) For any two complex numbers z_1 and z_2, prove the triangular inequality $|z_1 + z_2| \leq |z_1| + |z_2|$. When does the equality hold ? **14**

(ii) Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$.

OR

(b) (i) If f(z) has a finite limit at z_0, then f(z) is a bounded function in some neighborhood of z_0. Also verify the inequality $\sqrt{2}\,|z| \geq |\text{Re } z| + |\text{Im} z|$.

(ii) If the function g(z) is continuous at $z = z_0$ and the function f(z) is continuous at $g(z_0)$, then the composite function $f[g(z)]$ is continuous at z_0.

(c) Attempt any **four :** **4**

(i) Describe the region $\{z : \text{Im}(z) < |z - 1|^2\}$. State whether the region is a domain.

(ii) State the definition of connected set.

(iii) Compute the limit of function $\lim\limits_{z \to -i} \dfrac{(iz^3 + 1)}{z^2 + 1}$ if it exists ?

(iv) If $z_1 = -5$ and $z_2 = -1 + i$, show that $\text{Arg}(z_1 / z_2) = \text{Arg}(z_1) - \text{Arg}(z_2)$.

(v) Where the function $\text{Ln}(1 + z^2)$ is not continuous on the complex plane ?

(vi) State the definition of Uniform continuity of a complex function. Is $f(z) = 3z - 2$ is uniformly continuous in the region $|z| \leq 1$?

2. (a) (i) State and prove the sufficient condition for a function to be analytic. **14**

(ii) Show that the function Ln z is analytic for all z except when Re $z \leq 0$.

OR

(b) (i) Let f(z) be an analytic function on a connected open set D. If there are two constants c_1 and $c_2 \in C$, not all zero, such that $c_1 f(z) + c_2 \overline{f(z)} = 0$ for all $z \in D$, then show that f(z) is a constant on D.

(ii) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u + v = (x + y)(2 - 4xy + x^2 + y^2)$ then find u, v and the analytic function f(z) in terms of z.

(c) Attempt any **four :** **4**

(i) Are functions f(z) and $f(\bar{z})$ simultaneously analytic ? If yes, why ?

(ii) Let f(z) be an analytic function in a domain D. f(z) is constant in D if f'(z) vanishes identically in D. Is this statement true ? If yes, why ?

(iii) Is the function $w = |z|^2$ continuous everywhere but nowhere differentiable except at the origin ? If yes, why ?

(iv) State Cauchy-Riemann equations in polar form.

(v) The function $f(z) = z \sec z$ is not analytic at the points z = __________

(vi) If $u = x^2 - y^2$ is harmonic, then $\dfrac{\partial^2 u}{\partial z \partial \bar{z}}$ = __________

3. (a) (i) State and prove the existence of the contour integral. **14**

(ii) Evaluate the integral $I = \int_C (x + y^2 - ixy)dz$, where

$$C : z = z(t) = \begin{cases} t - 2i, & 1 \leq t < 2, \\ 2 - i(4 - t), & 2 < t \leq 3. \end{cases}$$

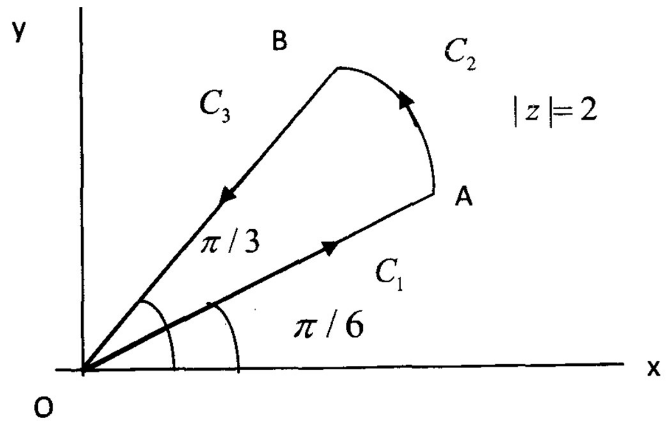
OR

(b) (i) Find a simple, piecewise smooth curve in t the form $z = z(t)$, whose trace and direction is given by the square with vertices at $z = 3 \pm 3i$ and $z = -3 \pm 3i$ traversed counter clockwise. Also check whether the curve

$$z(t) = \begin{cases} t + i(1 + t^2), & -2 \le t \le 1 \\ 2 - t + i[3 - (t-2)^2], & 1 < t \le 4. \end{cases}$$

is closed, simple, smooth or piecewise smooth.

(ii) Find an upper bound for the absolute value of the integral $I = \int_C (\exp(2z) - z^2)dz$, where C is the contour given in the figure below :



(c) Attempt any **three :** **3**

(i) The primitive of a function is unique up to an additive constant, why ?

(ii) A simple closed Jordan curve divides the Argand plane into _____ open domains which have the curve as common boundary.

(iii) Compute the length of the curve $z(t) = (1 - i)\exp(-it)$, $0 \le t \le \pi/2$.

(iv) Find an upper bound for the absolute value of the integral $I = \int_C \exp((\bar{z})^2)dz$, $C:|z|=2$ where C is traversed in the anticlockwise direction.

4. (a) (i) State and prove Cauchy integral theorem. **14**

(ii) Evaluate the integral $I = \oint_C \dfrac{\exp(z)}{z^2(z+1)^3}dz$, $C : |z| = 2$.

OR

(b) (i) State and prove Liouville's theorem.

(ii) Verify that the maximum and minimum modulus theorems hold for the function $f(z) = z^2 + 1$, where C is the circle $|z| = 1$ and D is the domain inside C.

(c) Attempt any **three :** **3**

(i) State Cauchy inequality.

(ii) If f(z) is an analytic function within and on a simple closed contour C and z_0 is any point inside C, then show that $\oint_C \frac{f(z)}{(z-z_0)^2} dz = \oint_C \frac{f'(z)}{(z-z_0)} dz$.

(iii) Evaluate the integral $I = \oint_C \frac{\exp(z)}{z^2 + 9} dz$, $C : |z| = 2$.

(iv) Define Simply connected domain. Can we convert Multiply connected domain into Simply connected domain ? If yes, how ?