Seat No. : _______________

AC2-04

April-2018

B.Sc., Sem.-VI

CC-309 : Mathematics

(Analysis – III)

Time : 3 Hours] **[Max. Marks : 70**

Note : (1) **All** questions are compulsory.
(2) Write the question number in your answer sheet as shown in the question paper.
(3) Figures to the right indicate marks of the question.

1. (a) Prove that every convergent sequence is a Cauchy sequence. Justify the converse of this theorem. **7**

OR

Let X be a metric space. Then prove that
(1) any intersection of closed sets in X is closed.
(2) any finite union of closed sets in X is closed.

(b) If the mapping d : R × R → R be defined by $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$, then prove that d is a metric on R. **7**

OR

If $\overline{A \cap B} = \bar{A} \cap \bar{B}$ is true then prove it and if not then give example violating this result.

2. (a) Prove that closed subsets of compact sets are compact. **7**

OR

Let (X, d_x), (Y, d_y) and (Z, d_z) be the metric spaces. If $f : E \subset X \to Y$ is continuous function at point $p \in E$ and function $g : Y \to Z$ is continuous function at point $f(p) \in Y$, then prove that function $gof : X \to Z$ is continuous function at point $p \in E$.

(b) The function $f : (0, 1) \to R$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. **7**

OR

Let A and B be two disjoint closed subset of metric space X, then prove that A and B are separated sets.

3. (a) Suppose that $\lim_{n \to \infty} f_n(x) = f(x)$ for $x \in E$ and let $M_n = \text{Sup}_{x \in E} |f_n(x) - f(x)|$. Then prove that $f_n \to f$ uniformly of E if and only if $\lim_{n \to \infty} M_n = 0$. **7**

OR

Let (f_n) be sequence of real or complex valued functions with domain E. Then prove that the (f_n) is uniformly convergent if and only if (f_n) is a Cauchy sequence.

(b) Give an example to show that a convergent series of continuous function have a discontinuous sum. **7**

OR

Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$, $x \in [0, 1]$, then prove that the sequence $f_n(x)$ is uniformly bounded but not uniformly convergence on [0, 1].

4. (a) State and prove Abel's limit theorem. **7**

OR

If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence r, then prove that the power series $\sum_{n=0}^{\infty} n a_n z^{n-1}$ and $\sum_{n=0}^{\infty} \dfrac{a_n}{n+1} z^{n+1}$ has also radius of convergence r.

(b) Show that for every $x \in R$, **7**

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$$

OR

Show that the function $f(x) = \begin{cases} e^{\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$ has derivatives of all orders at all $x \neq 0$ but does not have a Taylor's theorem.

5. Give the answer in brief : (any **seven**) **14**

(1) Define : Open Sphere in metric space.

(2) Give an example of a set which is both open and closed.

(3) Find the closure of the set of rational number Q and the set of real numbers R.

(4) Show that (0, 1) is not compact.

(5) Prove that $\sin^2 z + \cos^2 z = 1$, $z \in \mathbb{C}$.

(6) State Bolzano Weierstrass Theorem for metric spaces.

(7) State Binomial Series.

(8) Define : Uniformly continuous function.

(9) Define : Complete metric space.