

Seat No. : _____

SI-132

September-2020

B.Sc., Sem.-VI

**CC-307 : Mathematics
(Abstract Algebra-II)**

Time : 2 Hours]

[Max. Marks : 50

- Instructions :**
- (i) Attempt any **three** questions in Section-I.
 - (ii) Section-II is a compulsory section of short questions.
 - (iii) Notations are usual everywhere.
 - (iv) The right hand side figures indicate marks of the sub-question.

SECTION – I

Attempt any **THREE** of the following questions :

1. (a) Define a ring. Also prove the following properties in a ring R :
 - (1) $a \cdot 0 = 0 \cdot a = 0, \forall a \in R$, where 0 is the zero element of R .
 - (2) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b), \forall a, b \in R$. 7
- (b) Show that the set $Z(\sqrt{2}) = \{a + b\sqrt{2} / a, b \in Z\}$ forms a ring under usual addition and multiplication of real numbers. 7
2. (a) Prove that every field is an integral domain.
Also give an example of an integral domain which is not a field. 7
- (b) Define a Boolean ring and prove that a Boolean ring is a commutative ring.
Also give an example of a Boolean ring. 7

3. (a) Define an ideal of a ring R . Also prove that a nonempty subset I of a ring R is an ideal of R if and only if (i) $a - b \in I$, for all $a, b \in I$ and (ii) $a \cdot r$ and $r \cdot a \in I$, for all $a \in I$ and for all $r \in R$. 7
- (b) Show that $(\mathbb{Z}, +, \cdot)$, the ring of integers is a principal ideal ring. 7
4. (a) Prove that a field has no proper ideal. 7
- (b) Define a ring Homomorphism. If $\Phi : (R, +, \cdot) \rightarrow (R', \oplus, \odot)$ is a ring homomorphism and I is an ideal of R then prove that $\Phi(I)$ is an ideal of $\Phi(R')$. 7
5. (a) For nonzero polynomials $f, g \in D[x]$ prove that $[fg] = [f] \cdot [g]$. 7
- (b) Using Division algorithm for $f(x)$ and $g(x) \in Z_5[x]$ express $f(x)$ into the form $q(x)g(x) + r(x)$ for $f(x) = x^4 + 3x^2 + 2x + 4$ and $g(x) = x + 1 \in Z_5[x]$. 7
6. (a) Suppose $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \in Z[x]$ and suppose $\frac{p}{q}$ in the simplest form (i. e. $(p, q) = 1$) is a solution of the equation $f(x) = 0$. Then prove that $p|a_0$ and $q|a_n$. 7
- (b) Show that the polynomial $x^3 + 3x^2 - 8$ is irreducible over \mathbb{Q} . 7
7. (a) If \oplus and \odot are binary operations defined on the set \mathbb{R} of all real numbers as $a \oplus b = a + b - 1$; $a \odot b = a + b - ab$, then show that $(\mathbb{R}, \oplus, \odot)$ is a field. 7
- (b) If F_1 and F_2 are subfields of a field F , then prove that $F_1 \cap F_2$ also is a subfield of F . 7

8. (a) If M is a maximal ideal of a commutative ring R with unity then prove that the quotient ring R/M is a field. 7
- (b) If $I = \langle 4 \rangle$ then show that I is a maximal but not a prime ideal of the ring $2\mathbb{Z}$ of all even integers. 7

SECTION – II

9. Attempt any **FOUR** of the following in short : 8
- (i) Give an example of a division ring which is not a field.
 - (ii) Give an example of a subring which is not an ideal.
 - (iii) Give an example of a subring of a ring which is not an ideal of the ring.
 - (iv) Give an example of a division ring which is not a field.
 - (v) State the remainder theorem and the factor theorem for polynomials.
 - (vi) Define a prime ideal and give an example of a prime ideal.
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